

HUTP-01/A003
hep-th/0101218

Brane/anti-Brane Systems and $U(N|M)$ Supergroup

Cumrun Vafa

Jefferson Physical Laboratory
Harvard University
Cambridge, MA 02138, USA

Abstract

We show that in the context of topological string theories N branes and M anti-branes give rise to Chern-Simons gauge theory with the gauge supergroup $U(N|M)$. We also identify a deformation of the theory which corresponds to brane/anti-brane annihilation. Furthermore we show that when $N = M$ all open string states are BRST trivial in the deformed theory.

January 2001

1. Introduction

Topological strings propagating on Calabi-Yau threefolds provide a consistent vacuum of bosonic string theory [1] similar, and in some cases equivalent, to non-critical vacua of the bosonic strings. For example the B-model topological string on the conifold is equivalent to non-critical bosonic string propagating on a self-dual circle [2].

One can also consider open string version of these theories [3] which can be interpreted, in modern terminology, as adding D-branes to the Calabi-Yau. For example, if one considers N D-branes wrapping a Lagrangian 3-manifold L inside a Calabi-Yau, then the topological A-model gives rise to an open string field theory in the target space which is equivalent to ordinary Chern-Simons theory on L . In particular if we consider N D-branes on L we obtain $U(N)$ Chern-Simons theory on L (similarly in the B-model version one can consider D-branes wrapping holomorphic cycles and one ends up with a holomorphic version of Chern-Simons theory as the open string field theory).

The aim of the present note is to generalize these constructions to also include “anti-D-branes”. We find that, with a suitable notion of what “anti-D-branes” are in the topological theory, the theory involving N D-branes and M anti-D-branes wrapping the three manifold L in the context of A-type topological strings, leads to a Chern-Simons open string field theory with the gauge groups being the supergroup $U(N|M)$ (and similarly for the B-type case which leads to holomorphic Chern-Simons theory with supergroup connection). We also discuss the brane/anti-brane annihilation by giving a vev to a scalar field in the theory (which is the topological analog of “tachyon condensation”) and show how the physical states of the open string theory are removed upon this deformation.

Certain aspects of brane/anti-brane systems in the context of topological string theory and in particular its relation (in the context of B-model topological strings) with derived category has been pointed out in [4].

2. Review of Chern-Simons Theory as a String Theory

Consider A-model topological strings propagating on Calabi-Yau threefold, together with a Lagrangian submanifold L . Putting N D-branes on L we end up with the open topological string sector involving $U(N)$ Chern-Simons gauge theory [3],

$$S = \frac{ik}{4\pi^2} \int_L \text{Tr} \left[\frac{1}{2} A dA + \frac{1}{3} A^3 \right]$$

where the level ik (or $i(k + N)$ after quantum corrections) of the Chern-Simons theory is identified with the inverse of the string coupling constant. In other words the open string worldsheets in this context are exactly the 't Hooft diagrams of the $U(N)$ Chern-Simons theory.

The only physical string mode in the open string sector is the 1-form connection A and none of the oscillator modes of the open string theory are physical. In particular the general open string Chern-Simons field theory coincides with the ordinary Chern-Simons theory in this case. Moreover the BRST operator of the string Q is identified with the d operator

$$Q \leftrightarrow d$$

The space of vacua is parameterized by flat connections $F = 0$. For each such vacuum, corresponding to connection A_0 the BRST cohomology gets deformed to the covariant connection

$$Q \leftrightarrow d_{A_0} = d + [A_0, \cdot]$$

which can be readily seen by expanding the action near the new vacuum up to quadratic terms in the fluctuation field δA . For each such vacuum the physical states of the theory are in one to one correspondence with Q cohomology, which is the same as the tangent space to the moduli space of flat connections at $A = A_0$. Note that if there is no continuous family of flat connections, as would be the case if $\pi_1(L)$ is empty, there would be no physical state in the open string sector. This does not necessarily imply that open string sector gives zero partition function. For example if we take $L = S^3$ then there are no (local) physical states as the fundamental group of S^3 is trivial, but nevertheless the partition function of $U(N)$ Chern-Simons is highly non-trivial (in other words the worldsheet diagrams with boundaries do not give zero). There are in addition non-local observables for Chern-Simons theory, such as Wilson lines along links.¹

The open string field theory comes in general equipped with the ghosts which provide the BRST fields for the open string field theory. This is also the case in this particular example. The ghosts of the Chern-Simons gauge theory in this case correspond to enlarging A to be a combination of all forms [3][5], with even forms being fermionic and odd forms being bosonic. The ghost number of the string field theory coincides with the degree of

¹ It might be interesting to study the analog of Wilson loop observables for ordinary open string field theory which is also abstractly a Chern-Simons theory (in Witten's formulation).

the form. The usual physical field corresponds to ghost number 1, which is the one-form connection. However in topological string theory it is natural to consider all ghost number fields (including their zero modes) on equal footing and consider the extended theory. In particular the bosonic fields of this extended theory can in general have a bigger moduli space than the original theory as is well known in the context of closed topological strings (where it is called the extended moduli space). We will consider this extended theory also in the context of open strings and this will play an important role when we discuss brane/anti-brane annihilation below.

One can also consider the B-model topological strings and in this case (if we consider N wrapped D6 branes) we obtain a holomorphic version of Chern-Simons theory, whose basic field is a $(0, 1)$ -form connection \overline{A} of a holomorphic bundle, with the action

$$S = \frac{ik}{4\pi^2} \int \Omega \wedge Tr \left[\frac{1}{2} \overline{A} \partial \overline{A} + \frac{1}{3} \overline{A}^3 \right]$$

where Ω is the holomorphic 3-form on the Calabi-Yau. Here $Q = \overline{\partial} + [\overline{A}, \cdot]$. Similarly on lower even dimensional branes one obtains the natural reduction of the above action to the corresponding dimension.

3. Topological Anti-D-branes

We should decide what in the topological theory should distinguish branes from anti-branes. A natural choice, which we will adopt, is the following: The anti-branes should carry the opposite number of branes compared to branes. In the topological theory the only signature of dealing with a number of branes is the Chan-Paton factor N that one associates to each worldsheet boundary ending on them. Thus if we have M anti-branes, it is natural to associate a factor of $-M$ for each such hole. Put differently, for each anti-brane we put an extra minus sign for each worldsheet hole ending on it. This is our definition of topological anti-D-branes. Note that if we have only anti-branes wrapping some Lagrangian 3-cycle, the net effect of this on the partition function is the same as weighing worldsheet diagrams with odd number of holes with an extra minus sign. This in turn can be viewed as replacing the string coupling by minus itself, or replacing $ik \rightarrow -ik$ (or in the quantum corrected version $i(k+N) \rightarrow -i(k+N)$), which is the same as complex conjugation of the partition function. This is consistent with what one might naturally expect for anti-branes.

In the next section, we use the above definition of anti-branes and consider a situation where we have both branes and anti-branes and write down the effective open string field theory.

4. Brane/anti-Brane Systems and Chern-Simons Gauge Theory with Supergroup $U(N|M)$

Consider N branes and M anti-branes wrapped around a Lagrangian 3-cycle L . We would like to know what is the open string field theory for this theory. The physical state of the open string sector can be deduced, as usual, by considering the annulus diagram. In this case there are four such diagrams; One coming from boundaries ending on branes carrying a factor of N^2 —the corresponding physical state (at ghost number one) is naturally interpretable as the one-form connection in a $U(N)$ gauge group. The other diagram comes from both boundaries ending on an anti-Brane. This gives a factor of $(-M)(-M) = +M^2$, whose physical field can be interpreted as a one form connection for the gauge group $U(M)$. We also have two more diagrams ending on opposite type of branes, each giving a factor $N(-M) = -NM$. The corresponding physical states can be interpreted as 1-form gauge fields which are in the representation

$$(N, \overline{M}) \oplus (\overline{N}, M)$$

However, the extra minus sign in this annulus diagram implies that they are to be viewed as fermionic fields. In fact these fields naturally assemble themselves into the connection 1-form \mathcal{A} for a supergroup $U(N|M)$. It is also similarly straightforward to derive the open string field theory. It is the Chern-Simons gauge theory for the supergroup $U(N|M)$:

$$S = \frac{ik}{4\pi^2} \int_L Tr[\frac{1}{2}\mathcal{A}d\mathcal{A} + \frac{1}{3}\mathcal{A}^3]$$

where the notion of trace in the above action is in terms of the natural trace for the supergroup $U(N|M)$. The appearance of ‘superconnection’ is also natural from the viewpoint of brane/anti-brane systems in ordinary superstring theory [6][7][8][9]. However, in the superstring context the off-diagonal elements that appear in the superconnection are not connections (i.e. are not 1-forms), but are replaced by tachyons. Thus the superconnection in the usual superstring context, unlike here, cannot be viewed as a connection for a supergroup². In the next section we will discuss, in the topological context, how the analog of tachyon field arises.

² It would be interesting to see if there is any sense in which a gauge supergroup also exists in the superstring context.

Let us note that the partition function of the $U(N|M)$ Chern-Simons theory (expanded near the trivial connection) is the same as that of $U(N - M)$. This follows from the simple observation [10] that in the ‘t Hooft expansion of $U(N|M)$ gauge theory, the N, M dependence for each ‘t Hooft organization of Feynman diagram arises from boundary Chan-Paton factors, and for each boundary we obtain a supertrace over the fundamental representation of $U(N|M)$ which gives $(N - M)$. This is of course manifest from our definition of what anti-branes are. Thus the amplitudes for the Chern-Simons theory with supergroup $U(N|M)$ and that with gauge group $U(N - M)$ coincide, as one might expect for a theory with $N - M$ net number for branes. This again lends further support for the above proposal of what anti-branes are. In fact it has already been noted that also certain knot invariants for $U(N|M)$ theory are the same as that of $U(N - M)$ theory [11][12], though some subtleties arise when $N = M$ [13]. Apriori the above argument about ‘t Hooft diagram only implies the equality of partition functions and it is perfectly consistent with existence of certain observables of $U(N|M)$ theory which are not present in the $U(N - M)$ theory. Also if we expand about certain connections where the connection in $U(M) \subset U(N)$ are not equivalent, apriori there is no reason for the equivalence even of the partition function, with that of $U(N - M)$. This would then correspond to a situation where the branes and anti-branes cannot perfectly cancel one another even if the “tachyon mode” is turned on.

Note that the above discussion naturally generalizes to the case of topological B-models where one ends up getting holomorphic Chern-Simons theory with supergroup $U(N|M)$. Also one can formulate the reduction of this to lower dimensional holomorphic branes, and for example, study the topological version of the D0 branes dissolving in D2 branes.

5. Topological Analog of Brane/anti-Brane Annihilation

It is natural to ask if there is any topological analog of tachyon condensation and brane/anti-brane annihilation along the lines proposed by Sen [14]. At first the answer might appear to be in the negative, because the field which would have played the role of the tachyon is fermionic 1-form and it cannot take a vev. However upon closer inspection, as we will now argue the “tachyon field” has a more subtle presence in the Chern-Simons theory. We will identify a field whose vev plays the role of the vev for the tachyon field.

However, there is no potential for the scalar field and in particular it is not tachyonic in the usual sense.

For the case at hand, namely the Chern-Simons gauge theory with supergroup $U(N|M)$, just as in the case of ordinary gauge group $U(N)$, the open string field theory including the ghosts needed for BRST gauge fixing, leads to replacing \mathcal{A} by an arbitrary degree form, with alternating statistics. For the diagonal blocks of $U(N|M)$ the even forms are fermionic and odd forms are bosonic, whereas for off-diagonal block the odd forms of \mathcal{A} are fermionic and the even forms are bosonic. We will consider this extended theory, and treat all components of the extended \mathcal{A} field on the same footing. In particular there are *scalar* bosonic ghost fields in the off-diagonal blocks. These fields will play the role of the “topological tachyon fields”. In fact we will now demonstrate that turning them on is allowed (i.e. can lead to new classical solutions) and in case of $N = M$ and when the branes have the same gauge field configuration, trivializes the string observables.

Even though we have scalar analogs of tachyon fields, there are no scalar potentials for them. However we can still ask whether there are any other vacua for Chern-Simons theory, by turning them on³. We would like to identify the Chern-Simons analog of other critical points of tachyon potential. This means finding other classical solutions for the open string field theory. To satisfy the equations of motion, is equivalent to finding a vacuum which has a BRST operator Q which squares to zero: $Q^2 = 0$. As discussed before, this is equivalent to considering gauge field configurations where the curvature is zero, i.e.,

$$Q^2 = 0 \rightarrow d_{\mathcal{A}}^2 = (d + [\mathcal{A}, \cdot])^2 = 0 \rightarrow \mathcal{F} = 0$$

Note that \mathcal{F} is the curvature of the connection for the supergroup $U(N|M)$. Moreover, if we allow the bosonic ghosts also to take vev, this is equivalent to allowing the connection to have vevs for the odd forms in the diagonal blocks, and vevs for the even forms in the off-diagonal blocks.

Let us give an example of how this works: Consider two gauge field configurations for the bosonic parts of the supergroup $U(N) \times U(M)$, characterized by connections $A =$

³ Strictly speaking this makes sense in non-compact situation, where constant mode fluctuation is not allowed. We can assume we are in the non-compact case, for example when $L = R^3$ in a non-compact CY such as \mathbf{C}^3 —but one can also make sense of it in the compact case as field configurations which satisfy the equations of motion, by first integrating over the non-zero modes in the path-integral.

$A_N + A_M$. Suppose there is a covariantly constant field ϕ in the bifundamental (N, \overline{M}) , i.e.,

$$d_A \phi = d\phi + A_N \phi - \phi A_M = 0$$

If we give a vev to this off diagonal component of $U(N|M)$ at degree 0, i.e. the “bosonic ghost field”, then we have another vacuum for the Chern-Simons theory with the gauge supergroup. In particular expanding to quadratic order, we can identify the new BRST operator

$$Q = d_{\mathcal{A}} = d_A + [\phi, \cdot]$$

where written in this form we view ϕ as an upper block off-diagonal matrix valued element of $U(N|M)$ adjoint representation. Note that $Q^2 = 0$ and we can thus view this as a new vacuum of the theory. This is also consistent with the viewpoint advocated in [4] for cohomological aspects of brane/anti-brane annihilation and its relation to derived category.

The physical states in the new vacuum will in general be different, because now we have a different cohomology problem defined by Q . Let us specialize to the following case: Suppose we have equal number of branes and anti-branes $N = M$ and assume we have the same flat $U(N)$ gauge connection on both types of branes. Then we have a natural choice for ϕ , namely the identity $N \times N$ matrix. We will identify giving a vev to this field as “brane/anti-brane annihilation”, which is the analog of tachyon condensation in this topological setup. As a test of this idea we show that the open string field theory BRST operator Q for this new vacuum after giving a vev to ϕ trivializes the cohomology elements of the old Q . In particular suppose we originally have a number of deformations for the flat bundle for both $U(N)$ ’s, which by our assumption they are the same. Restricted to each of these modes the new Q operator will reduce to $Q = [\phi, \cdot]$. It is easy to see that with respect to an upper block off-diagonal identity matrix ϕ all the cohomology disappears: We can view the $(N|N)$ adjoint as consisting of four natural $(N \times N)$ blocks. We now show that the new Q pairs up these four blocks. The lower off diagonal block is not Q invariant (as it does not commute with ϕ). It instead gives the equal sum of the two diagonal blocks. On the other hand Q acting on the opposite combination of the two diagonal blocks will give all upper block off-diagonal combinations. Thus all the cohomologically non-trivial elements are now paired up and we have no non-trivial cohomology left, as expected for a theory which is to be identified with a trivial theory. Note that we have already explained why the partition function will be trivial, when the brane/anti-brane bundles are expanded near the trivial connection. That argument can be extended to the case where the connections

in the two $U(N)$'s are the same. But consider a situation where L has a non-trivial π_1 . In that case we can consider inequivalent configurations for the two gauge groups, and from the perspective of 't Hooft diagram, there is no reason why such configurations should give vanishing contribution to the partition function. It is amusing that precisely for these cases there is no covariantly constant identity matrix ϕ (i.e. precisely for the case where we do not expect complete annihilation there is also no reason for the partition function to vanish).

6. Potential Applications to Superstring Compactifications

As discussed in [15], topological strings with branes wrapped over Lagrangian submanifolds compute certain terms in the effective theory of type IIA strings compactified on Calabi-Yau manifolds with partially wrapped D6 branes, filling the spacetime. Also, as discussed there, in the context of compact Calabi-Yau we have to consider equal number of branes and anti-branes, and for some such cases, a large N dual was proposed. Given that we have defined what topological superstrings are in the context of branes and anti-branes, it is natural to ask if there are similar terms in the type IIA superstring context with wrapped branes and anti-branes that they compute. This of course would sound very remarkable given that this would correspond to some exact computations in a non-supersymmetric background. However, this may not be as surprising. For example, the net number of branes is a computation which one expects to be able to do exactly in the non-supersymmetric cases. Also superpotential terms are another example of what branes and anti-branes give rise to that should be exactly computable (similar to turning on RR fluxes in Calabi-Yau, which for a generic moduli is not supersymmetric, but give rise to computable superpotential terms). The terms that the topological superstrings compute in the cases including branes and anti-branes must be generalizations of such terms and it would be very interesting to precisely identify them.

I would like to thank the hospitality of the string theory group at Rutgers University. In particular I have greatly benefitted from discussions with M. Douglas, M. Marino and G. Moore. I would also like to thank N. Berkovits, K. Hori, S. Katz and I.M. Singer for valuable discussions.

This research is supported in part by NSF grants PHY-9802709 and DMS 9709694.

References

- [1] E. Witten, “On the structure of the topological phase of two dimensional gravity,” Nucl. Phys. **B340** (1990) 281.
- [2] D. Ghoshal and C. Vafa, “ $c=1$ string as the topological theory of the conifold,” Nucl. Phys. **B45** (1995) 121.
- [3] E. Witten, “Chern-Simons theory as a string theory,” hep-th/9207094.
- [4] M. Douglas, “D-branes, categories and $N=1$ supersymmetry,” hep-th/0011017.
- [5] S. Axelrod and I.M. Singer, “Chern-Simons perturbation theory–II,” J. Diff. Geom. **39** (1994) 173.
- [6] E. Witten, “D-branes and K-theory,” JHEP9812 (1998) 019.
- [7] P. Kraus and F. Larsen, “Boundary string field theory of the DDbar system,” hep-th/0012198.
- [8] T. Takayanagi, S. Terashima and T. Uesugi, “Brane-Antibrane action from boundary string field theory,” hep-th/0012210.
- [9] M. Alishahiha, H. Ita and Y. Oz, “On superconnections and the tachyon effective action,” hep-th/0012222.
- [10] N. Berkovits, C. Vafa and E. Witten, “Conformal field theory of AdS background with Ramond-Ramond flux,” JHEP9903 (1999) 018.
- [11] J.H. Horne, “Skein relations and Wilson loops in Chern-Simons gauge theory,” Nucl. Phys. **B334** (1990) 669.
- [12] M. Bourdeau, E.J. Mlawer, H. Riggs and H.J. Schnitzer, “The quasirational fusion structure of $SU(M|N)$ Chern-Simons and W-Z-W theories,” Nucl. Phys. **B372** (1992) 303.
- [13] L. Rozansky and H. Saleur, “Reidemeister torsion, the Alexander polynomial and $U(1,1)$ Chern-Simons theory,” J. Geom. Phys. **13** (1994) 105.
- [14] A. Sen, “Tachyon condensation on the brane antibrane system,” JHEP 9808 (1998) 012.
- [15] C. Vafa, “Superstrings and topological strings at large N ,” hep-th/0008142.